was performed only on the nonads from the fifth to the eleventh, inclusive, since the first four sums-of-products would have reproduced a part of s_{156}' and not of r_{156}. (All of the nonads after the sixth were composed entirely of supplied zeros both for N and q_{156}.) The last figures of s_{156}' were copied on the third line from the bottom of the fifth strip so that by forming the difference between the second and third lines the required value of r_{156} came out on the bottom line. This remainder was found on Aug. 11, 1944, to be 118 57508 80382 71696 98184 73569 85091 23773 18030 92037. Since this residue is not zero it follows that M_{157} is composite and incidentally that M_{127} still retains the position of being the largest known prime number. For every value of k from 8 to 156 the numbers on the three corresponding strips were found to satisfy the relation $s_k' = N \cdot q_k + r_k$ for each of the moduli $10^3 + 1$, $10^4 + 1$ and $10^7 + 1$. The author desires to announce that he has already begun to investigate M_{167}.

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*ON THE STABILITY OF TWO-DIMENSIONAL PARALLEL FLOWS**

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1. *Introduction.*—Heisenberg's[1] remarkable contribution to the hydrodynamic stability of two-dimensional parallel flows has not been favorably accepted and properly appreciated, because his paper is not completely free from obscure points. Nor has the work of Tietjens,[2] Tollmien[3] and Schlichting[4] been properly estimated. As a result, the theory to account for the instability of laminar flow at high Reynolds numbers has become very confused, and its further development has been very much retarded. Various authors suggest that it is necessary (1) to consider disturbances of finite amplitudes, (2) to include the effect of compressibility or even (3) to modify the Navier-Stokes equations. The present situation of our knowledge may be seen from the general lectures given by G. I. Taylor[5] and J. L. Synge.[6]

Recently the present writer carried out some investigations in an attempt to clarify the situation. The theory of Heisenberg was critically examined, somewhat modified and further developed. These developments were made with particular emphasis on the general criteria of instability and their underlying physical mechanism, both in an inviscid fluid and in a real fluid. According to the general criterion obtained, the plane Poiseuille flow and the Blasius flow are both unstable at sufficiently high Reynolds numbers.[7] Detailed numerical calculations were also worked out in these cases. The present article gives a brief account of the essential results obtained. A detailed treatment will be published elsewhere.

2. The Equation of Orr and Sommerfeld.—In studying the stability of two-dimensional parallel flows, we may restrict ourselves to two-dimensional disturbances, according to an investigation of Squire.[8] A two-dimensional periodic disturbance in a field of flow with main velocity $w(y)$ parallel to the x-axis may be represented by the stream function $\psi = \varphi(y)e^{i\alpha(x-ct)}$, and the linearized differential equation for $\varphi(y)$ is

$$(w - c)(\varphi'' - \alpha^2\varphi) - w''\varphi = -\frac{i}{\alpha R}(\varphi^{iv} - 2\alpha^2\varphi'' + \alpha^4\varphi). \quad (2.1)$$

All the quantities involved are dimensionless: R is the Reynolds number of the main flow, and α, c are constants describing the nature of the disturbance. We shall regard α as real so that $2\pi/\alpha$ is the wave-length of the disturbance. The real part of c gives the phase velocity while its imaginary part (multiplied by α) gives the index of damping or magnification. The function $w(y)$ is regarded as defined for all complex values of y by analytic extension. Physical considerations then lead to certain homogeneous boundary conditions to be satisfied by $\varphi(y)$ at two real points y_1 and y_2 of the complex y-plane, which represent the coördinates of certain layers in the field of flow. There are altogether four boundary conditions. Hence, we are led to an eigenvalue problem, and a relation of the type

$$F(\alpha, c, \alpha R) = 0 \quad \text{or} \quad c = c(\alpha, R) \quad (2.2)$$

holds, where F is indeed an entire function of the parameters. Since stability, neutral stability or instability of the motion correspond to the real part of $-i\alpha c$ less than, equal to or greater than zero, the problem is to determine the imaginary part $c_i(\alpha, R)$ of $c(\alpha, R)$ for a set of real values of α and R.

The actual investigation depends upon the solution of (2.1) by the method of successive approximations. Two methods are used. First, we introduce the variable

$$\eta = (y - y_0)/\epsilon, \quad (2.3)$$

where $w(y_0) = c$, $\epsilon = (\alpha R)^{-1/3}$, and $w'(y_0)$ is positive for real values of c. In this way, we obtain a fundamental system of four solutions $\varphi(y)$ in the form

$$\varphi(y) = \sum_{n=0}^{\infty} \epsilon^n \chi^{(n)}(\eta), \tag{2.4}$$

where $\chi^{(n)}(\eta)$ are explicitly expressible in terms of the Hankel functions $H_{1/3}^{(1),(2)}[{}^2/_3(i\alpha_0\eta)^{3/2}]$, $\alpha_0 = \{w'(y_0)\}^{1/3}$. Indeed, the series (2.4) can be shown to be convergent for the range of variables and parameters concerned. Secondly, we may put

$$\varphi(y) = \sum_{n=0}^{\infty} (\alpha R)^{-n} \varphi^{(n)}(y), \tag{2.5}$$

which gives us two asymptotic solutions. The initial approximation of (2.5) satisfies the inviscid equation

$$(w - c)(\varphi^{(0)''} - \alpha^2\varphi^{(0)}) - w''\varphi^{(0)} = 0, \tag{2.6}$$

which can be solved by expanding $\varphi^{(0)}$ in power series of α^2, convergent for all values of α^2. Two other asymptotic solutions are obtained in the form

$$\varphi(y) = \exp\{\textstyle\int g dy\}, \qquad g = (\alpha R)^{1/2}g_0 + g_1 + (\alpha R)^{-1/2}g_2 + \ldots. \tag{2.7}$$

Both methods of solution are essentially due to Heisenberg,[1] and Tollmien followed the same way. Heisenberg also carried out some approximate calculation for the case of channel flow, while Tollmien studied the case of the boundary layer. Criticism of Heisenberg's work is usually associated with the multiple-valued nature of the asymptotic solutions and the convergence of the power series in α^2 used for solving (2.6). Detailed investigations justify Heisenberg's treatment. In fact, the asymptotic solutions hold only for certain regions of the y-plane to be determined by comparison with the asymptotic expansions of (2.4). Indeed, from the properties of the Hankel functions, it can be easily shown that under the restriction

$$-\frac{7\pi}{6} < \arg(\alpha_0\eta) < \frac{\pi}{6}, \tag{2.8}$$

the asymptotic expansions do not suffer any jump (which may be called the Stokes phenomenon or the Uebergangs-substitution). Fortunately, such a restriction does not exclude the existence of a connected region enclosing the points y_1 and y_2 where our boundary conditions are to be satisfied. The eigenvalue problem can therefore be conveniently treated with the help of the asymptotic solutions. However, the region of validity of the asymptotic expressions does not always include all the points of the real segment (y_1, y_2). As a matter of fact, the asymptotic solutions hold throughout the real interval (y_1, y_2) only when $c_i > 0$. For $c_i < 0$, there are two points on the real axis where the Stokes jump of the asymptotic solutions occur. These two points coalesce into one critical point when $c_i = 0$.

Physically, the failure of the asymptotic solutions at some points on the real axis (and therefore the failure of $\varphi^{(0)}$ as a first approximation) in the case of neutral ($c_i = 0$) and damped oscillations ($c_i < 0$) indicates the presence of the effect of viscosity at the corresponding layers inside the fluid. This was noted by several authors in the cases of neutral stability. However, the existence of *two inner viscous layers* for the damped disturbance does not seem to have been noticed before. As we shall see in Section 3, it has an important consequence upon the well-known criteria of Rayleigh[9] and Tollmien.[10]

3. General Criteria of Stability and the Curve of Neutral Stability.—In the limit of infinite Reynolds numbers, it can be shown by careful mathematical investigation that the problem can be treated by finding a solution of (2.6) satisfying two boundary conditions. Rayleigh[9] and Tollmien[10] have shown that the necessary and sufficient condition for the existence of neutral or self-excited disturbances is that the velocity curve has a point of inflection in the interval (y_1, y_2). The necessary condition is general, while the sufficient condition has been proved by Tollmien only for velocity distributions of the symmetrical type or of the boundary-layer type. In fact, an example has been shown by the present writer where a point of inflexion exists in the velocity curve, but a neutral disturbance does not exist.

The authors mentioned above also concluded that damped disturbances are excluded if a point of inflexion does not exist. However, their proof assumes the analytical nature of the solution $\varphi^{(0)}(y)$ along the real axis, which is true for self-excited disturbances, but not for damped disturbances. Previous authors regarded a damped disturbance as the complex conjugate of a self-excited disturbance. This is in contradiction with the above-mentioned drastic difference of nature of those solutions along the real axis. Careful investigation reveals that the damped or self-excited nature of a solution is not changed by taking the complex conjugate. *Therefore damped disturbances are not excluded by the condition* $w''(y) \neq 0$ (at least not according to Rayleigh's proof).

Tollmien's proof to establish the existence of a point of inflexion in the velocity curve as sufficient to insure instability is rather cumbersome, and he has to assume that w''' does not vanish at the point of inflexion. These points have now been improved by a more careful consideration of the analytical nature of (2.6).

Having settled the question of hydrodynamic stability at infinite Reynolds numbers, the author studied the stability in a real fluid by trying to find out how the foregoing results should be modified for large but finite values of the Reynolds number. One such result has already been obtained by Heisenberg,[1] whose conclusion is as follows. If a velocity distribution allows an inviscid neutral disturbance with finite wave-length and non-vanishing phase velocity, the disturbance with the same wave-length is unstable in the real fluid when the Reynolds number is sufficiently large. In the present investigation, the proof has been formulated in a slightly more satisfactory manner.

Since a point of inflection in the velocity curve is sufficient to insure instability only for velocity distributions of the symmetrical and boundary-layer types, the author has also made more attempts in these cases. Indeed, it has been shown that *in these cases the flow is always unstable for sufficiently large Reynolds numbers*, whether the velocity curve has a point of inflection or not. The curve of neutral stability

$$c_i(\alpha, R) = 0 \tag{3.1}$$

may belong to either of the types shown in figure 1. When the velocity curve has no point of inflection, the two asymptotic branches of the curve have the common asymptote $\alpha = 0$ (Fig. 1 *a*). When there is a point of inflection, one branch has the asymptote $\alpha = 0$, while the other has the asymptote $\alpha = \alpha_s > 0$ (Fig. 1 *b*). These results are in agreement with those of Rayleigh, Tollmien and Heisenberg cited above.

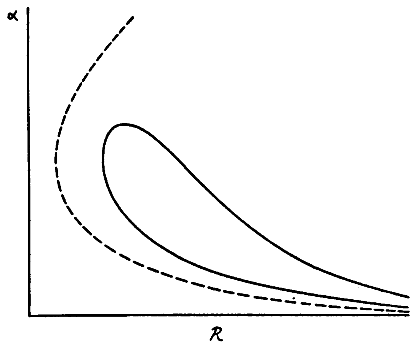

FIGURE 1 (a)

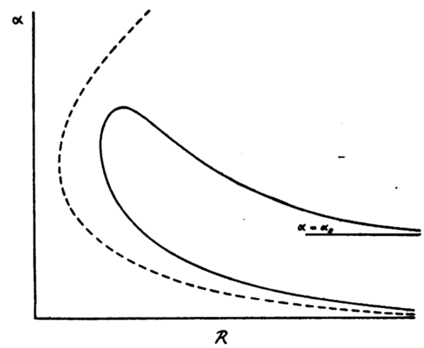

FIGURE 1 (b)

General nature of the curve of neutral stability. The dotted curve is a curve of stability given by Synge.

Simple general formulae have also been derived to express the exact forms of those asymptotic branches in terms of the velocity curve. The fact that these two asymptotic branches join together to give a maximum α and a minimum R (instead of going to infinite α) can be inferred from a criterion of Synge.[6] It states that there is always stability ($c_i < 0$) if

$$(qR)^2 < (2\alpha^2 + 1)(4\alpha^4 + 1)/\alpha^2, \quad q = \max_{y_1 \leq y \leq y_2} |w'(y)|. \tag{3.2}$$

Indeed, the minimum value of the Reynolds number on the neutral curve marks the beginning of instability and is therefore very important. Accordingly, the following approximate formulae have been derived for the evaluation of this minimum value:

$$R = \frac{30w_1'}{c^3}\left\{\frac{w_1'}{c}\int_{y_1}^{y_2} w^2 dy\right\}^{1/2}, \text{ for symmetrical profiles,} \tag{3.3}$$

$$R = \frac{25w_1'}{c^4}, \text{ for boundary-layer profiles,} \tag{3.4}$$

where w_1' is the value of $w'(y)$ at $w = 0$, and c is given by

$$-\pi w_1' \frac{w(y)w''(y)}{\{w'(y)\}^3} = 0.6 \text{ at } w(y) = c. \tag{3.5}$$

In these dimensionless expressions, the velocities are referred to the maximum velocity and the lengths to the distance from $w = 0$ to $w = 1$. Complete numerical calculation of the curve $c_i(\alpha, R) = 0$ has been carried out for the plane Poiseuille flow and the Blasius flow. The results are in general agreement with figure 1 (a). For the channel, the critical Reynolds number is found to be 16,000 as defined in terms of the maximum velocity and the width of the channel; for the boundary layer, it is found to be 400 as defined in terms of the free-stream velocity and the displacement thickness. The result in the latter case is in better agreement with Tollmien's. original calculation[3] than with Schlichting's later one.[4] The maximum value of α is even larger than Tollmien's value, while Schlichting's value is smaller.

4. *Physical Interpretations and Concluding Discussions.*—We shall now try to understand the physical mechanism underlying the above results. Generally speaking, the investigation in an inviscid fluid brings out the rôle of pressure and inertial forces. The subsequent investigation of viscous forces will then settle the stability problem of a real fluid.

Equation (2.1) is the equation of vorticity. This suggests that the condition $w'' = 0$ in Rayleigh-Tollmien's criterion means an extremum of vorticity of the main flow $w(y)$. In fact, the physical mechanism can be understood by following this line of thought. Let us regard the field of flow as vorticity distribution in parallel layers, and imagine a disturbance consisting of the interchange of two fluid filaments in two different layers. These fluid elements will retain their original vorticities, and will therefore be present in layers with vorticities different from their own. It can be shown that these filaments will then be pushed back to their original layers if the vorticity gradient of the main flow has the same sign.[12] The flow is therefore stable. Indeed, if we consider an excess concentrated distribution of vorticity $\zeta(x, y)$ (a "vortex") upon a small region of a given field of vorticity, $\zeta_0(y)$, then the vortex is accelerated with an average acceleration

$$\bar{a} = \frac{1}{\Gamma}\int\int v^2(x, y)\zeta_0'(y)dxdy \tag{4.1}$$

in the direction of the positive y-axis, where Γ is the total strength of the vortex, $v(x, y)$ is the y-component of velocity due to it, and the integration is taken over the whole field of flow. A consideration of the sign of Γ and $\zeta_0'(y)$ will then substantiate the above statements. Thus, the stability in an inviscid fluid as governed by inertial and pressure forces can be discussed in terms of the gradient of vorticity of the main flow.

Another interpretation of the significance of the gradient of vorticity is to be found from Kelvin's "cat-eye" picture of a flow with neutral disturbance.[18] If we consider the vorticity of fluid elements along a closed stream line in that picture, we see that such a flow pattern is not possible without diffusion of vorticity by viscous forces, if the gradient of vorticity is finite at the critical layer where $w = c$.

This consideration immediately leads to a simple method of visualizing the extent of viscous forces in controlling the stability of fluid motion. If we consider the diffusion of vorticity from the critical layer for a period T of a neutral disturbance, the effective distance covered would be of the order of $(\nu T)^{1/2}$, ν being the kinematical viscosity. The ratio of this distance to that between the critical layer and the solid boundary can be easily shown to be

$$s = \sqrt{\frac{2\pi}{z^3}} \tag{4.2}$$

where $z = -\alpha_0\eta_1$ is directly related to the argument of the Hankel functions of Section 2, η_1 being the value of η at $y = y_1$. This parameter s has the value 0.7 at infinite Reynolds number on the lower branch of the neutral curves of figure 1. It decreases to about 0.5 at minimum Reynolds number and further decreases to zero as R becomes infinite along the upper branch. Referring to those figures, we note that for small values of s (small viscosity), the effect of viscosity is essentially destabilizing, since an increase of Reynolds number gives more stability (cf. Heisenberg's criterion of Section 3). For large values of s (large viscosity), the opposite is true. The physical mechanism can be described in the following way. It is known that the destabilizing effect is caused by phase shifts of the disturbance, which tends to build up its shear $-\rho\overline{u'v'}$ (u', v' being the components of velocity of the disturbance, and ρ the density of the fluid). This, in turn, transfers the energy of mean flow into that of small oscillations. The stabilizing effect is due to dissipation. Now, for small values of s, we may think of the flow as having two thin viscous layers, one at the wall, the other at the layer where $w = c$; the effect of dissipation is relatively unimportant, and the resultant influence of viscous forces is destabilizing. When s exceeds 0.5, these two layers may be regarded as having joined each other into one viscous region; the effect of dissipation becomes important, and the resultant influence is stabilizing.

The viscous forces upset the decisive nature of an extremum of vorticity for the determination of the stability of the flow of a real fluid, as one can see from a comparison of figure 1 (*a*) (no extremum) and figure 1 (*b*) (with extremum). Thus, there is no drastic change in the stability characteristics of a boundary layer as the pressure gradient changes from a slightly positive value to a slightly negative value (as one would expect if one tries to apply Rayleigh-Tollmien's criterion to a real fluid). However, the appearance of the point of inflection in a velocity curve does indicate an increase of instability. It can be shown, by using (3.3), (3.4) and (3.5), that the critical Reynolds number will be decreased. Also, for a given Reynolds number, the range of wave-lengths of unstable disturbances will be increased. For a boundary layer with adverse pressure gradient, a further indication of increasing instability is given by the decrease of w_1'. Indeed, as $w_1' \to 0$, the approximate formulae (3.4), (3.5) give $c \to 1$, $R \to 0$. This is, of course, an extrapolation of those formulae, but the conclusion stands in qualitative agreement with the usual notion that the flow would be violently unstable in such a case (separation).

5. *Transition to Turbulence.*—Finally, we want to touch briefly the question of transition to turbulence. A linearized theory of hydrodynamic stability is of course not quite adequate to account for such a phenomenon. But if we allow our theory to be extrapolated to the beginning of non-linear effects, and assume the subsequent development into turbulence to be very rapid, there is the possibility of determining approximately the Reynolds number of transition. For flow in boundary layers, it is interesting to note that the disturbance in the free stream has an isotropic nature in two dimensions. There may be therefore the possibility of introducing the results of the theory of instability in boundary layers into Taylor's theory of the transition Reynolds number,[5] which appears to be quite successful when the transition is caused by the turbulence in the free stream.

The author wishes to express his sincere gratitude to Professor Theodore von Kármán for suggesting this problem and for his invaluable help throughout this work; to Professor Clark B. Millikan for helpful suggestions and for using some of his unpublished notes, from which much inspiration and suggestion has been derived; to Professors P. S. Epstein, H. Bateman, Dr. Hans W. Liepmann and several of his friends for many valuable discussions and suggestions.

* The present article is a brief report of some results contained in the author's dissertation approved by the California Institute of Technology for the degree of Doctor of Philosophy.

[1] Heisenberg, W., *Ann. d. Phys.*, **74**, 577–627 (1924), especially p. 599.

[2] Tietjens, O., *Zeitz. Angew. Math. Mech.*, **5**, 200–217 (1925).

[3] Tollmien, W., *Göttinger Nachrichten*, 21–44 (1929).

[4] Schlichting, H., *Ibid.*, 181–208 (1933).

[5] Taylor, G. I., *Proceedings of the Fifth International Congress for Applied Mechanics*, Cambridge, Mass., U.S.A. (1938), pp. 304–310, especially equation (45), p. 307.

[6] Synge, J. L., *Semicentennial Publications Amer. Math. Soc.*, Vol. II: Semicentennial Addresses, pp. 227–269 (1938); especially equation (11.23), p. 258.

[7] The conclusion that the plane Poiseuille flow may be unstable with respect to certain disturbances is in contradiction with that of Noether, F. (ZaMM, **6**, 232–243) (1926), and Pekeris, C. L. (*Jour. Aero. Sci.*, **5**, 237–240) (1938). The untenable points in their investigation are discussed in the present investigation.

[8] Squire, H. B., *Proc. Roy. Soc. London*, **A142**, 621–628 (1933).

[9] Rayleigh, Lord, *Scientific Papers*, Vol. 1, pp. 474–487.

[10] Tollmien, W., *Göttinger Nachrichten, Neue Folge*, **1**, 70–114 (1935).

[11] Rayleigh's treatment of this problem using broken linear velocity distributions has been denounced by Heisenberg, but his proof of the necessary condition is acceptable, except for the modification to be discussed below.

[12] The fact that an element with excessive vorticity is accelerated toward the region of higher vorticity was recognized and suggested to the author by Th. v. Kármán in connection with the theory of vorticity transfer in a fully developed turbulent flow.

[13] Kelvin, Lord, *Math. Phys. Papers*, Vol. 4, pp. 186–187.

FLAT SPACE-TIME AND GRAVITATION

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If one admits that physical events take place in a 4-dimensional space-time continuum (an idea abandoned in current quantum-mechanical theory) there are three interesting possibilities: classical space and time; flat or electromagnetic space-time; curved space-time. The appropriate corresponding mathematical languages are, respectively, those of 3-vectors, 4-vectors and 4-tensors.

In a certain sense the flat space-time, characteristic of the so-called special theory of relativity, is just as *absolute* as classical space and time, since the coördinates t, x, y, z require exactly 10 arbitrary constants for their complete specification in both cases. But, in the framework of flat space-time, the fundamental electromagnetic equations of Maxwell and Lorentz lose the artificiality which they possess in classical space and time.

The initial attempts to incorporate gravitational phenomena in flat space-time were not satisfactory. Einstein turned to the curved space-time suggested by his principle of equivalence, and so constructed his general theory of relativity. The initial predictions, based on this celebrated theory of gravitation, were brilliantly confirmed. However, the theory has